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Stabilization of Solutions to a FitzHugh-Nagumo Type System

Danielle Hilhorst · Piotr Rybka

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Abstract We consider a bistable reaction-diffusion system arising in the theory of phase transitions; it appears in several physical contexts such as thin magnetic films and the microphase separation in diblock copolymer melts. Mathematically it takes the form of an Allen-Cahn equation coupled to an elliptic equation. This system possesses a Lyapunov functional which represents the Gibbs free energy of the phase separation problem. We study the large time behavior of the solution orbits, and use the fact that the problem has a gradient structure to prove their stabilization by means of a version of Łojasiewicz inequality.

Keywords Gradient flow · Łojasiewicz inequality · Stabilization of solutions · FitzHugh-Nagumo system · Diblock copolymer · Infinite dimensional dynamical systems

1 Introduction

In this article we consider the following reaction-diffusion system of bistable type, where an Allen-Cahn equation is coupled to an elliptic equation

$$\begin{aligned} u_t &= D_u \Delta u + f(x,u) - v, \quad \text{in } \Omega \times (0,\infty), \\ 0 &= D_v \Delta v - av + \gamma u - b, \quad \text{in } \Omega \times (0,\infty), \end{aligned} \tag{1.1}$$

together with homogeneous Neumann boundary data for u and v and an initial condition for u. Here, a, b, γ, D_u and D_v are positive constants, and Ω is an open, bounded subset of $\mathbb{R}^n$, $n \geq 1$, with smooth boundary.

D. Hilhorst (✉)
CNRS and Laboratoire de Mathématiques, Université de Paris-Sud 11, Bât. 425, 91405 Orsay, France
e-mail: Danielle.Hilhorst@math.u-psud.fr

P. Rybka
Institute of Applied Mathematics and Mechanics, The University of Warsaw, ul. Banacha 2, 02-097 Warszawa, Poland
e-mail: p.rybka@mimuw.edu.pl

When $f(x,u) = f(u)$, with for instance $f(u) = u - u^3$, system (1.1) may be regarded as a special case of the FitzHugh-Nagumo reaction-diffusion system

$$\begin{aligned} u_t &= D_u \Delta u + f(u) - v, \quad \text{in } \Omega \times (0,\infty), \\ \tau v_t &= D_v \Delta v - av + \gamma u - b, \quad \text{in } \Omega \times (0,\infty), \end{aligned} \tag{1.2}$$

with $\tau = 0$.

The FitzHugh-Nagumo system arises in neuro-physiology. It is a simplified form of the Hodgkin-Huxley system which describes electronic and ionic events occurring during the transmission of an impulse along an axon, namely the filament carrying signals from the nerve cell body to other parts of the organism. Its formulation is based upon the assumption that an axon behaves like a cylindrical electrical cable with conducting core and partially insulation sheath [8, 16].

Problem (1.1) or closely related systems also appear in other physical contexts such as thin magnetic films [9] and the microphase separation in diblock copolymer melts. Let us comment on the last one. A diblock copolymer is a linear-chain molecule of two subchains jointed covalently to each other. Each subchain is made of different monomers. Below a critical temperature the subchains begin to segregate due to repulsion between unlike monomers.

The above problems are gradient flows and involve a free energy functional of nonlocal type [17, 18] and [21]. The Lyapunov functional $\mathcal{E}$, which is given by

$$\mathcal{E}(u) = \int_\Omega \left(\frac{D_u}{2} |\nabla u(x)|^2 + F(x,u(x)) \right) dx + \frac{1}{2} \int_\Omega |\nabla v(x)|^2 \, dx, \tag{1.3}$$

where $F(x,u) = -\int_0^u f(x,s)\,ds$, may represent the Gibbs free energy of a phase separation problem. The critical points of this variational problem can be regarded as the thermodynamic equilibrium states of the phase separation phenomenon.

In this paper, in Sect. 3, we study the large time behavior of the solutions of Problem (1.1). More precisely we show that any solution of (1.1) converges to a steady state. For this purpose we use the Łojasiewicz inequality (see [3, 13–15, 20] and references therein). This argument depends, in an essential way, on the analyticity of the nonlinear term f, which is here a polynomial in u of the third degree.

On the way, in Sect. 2, we prove again an existence result. In order to avoid unnecessary technical difficulties, we assume that the initial datum u_0 has already been smoothed out by the flow. Our tool is the standard theory of analytic semigroups, as exposed in Henry's book [12]. However, the key to obtain a global in time result is to perform a priori estimates on solutions. This clearly appears in the proof of Lemma 2.1 below.

The reason for using the argument based on analyticity stems from the fact that we do not fully know the structure of the steady states of (1.1). However, we know that in similar systems in $\Omega = (0,1)$ the number of equilibria is finite (see e.g. [10]), so that convergence to equilibria is a well-know property. Here, we concentrate on the case $n \geq 2$. We also note (cf. [3]) that if a solution converges to an isolated steady state, then we automatically obtain an exponential decay rate.

Our paper is one of the series of articles devoted to studies of stabilization of gradient-like systems whose main tool is Łojasiewicz inequality. The first was the paper by Łojasiewicz himself [13, 15], who showed that any bounded solution to gradient systems in $\mathbb{R}^n$ (which is an ODE system), converges to a stationary point. This idea was subsequently developed for infinite dimensions gradient systems by L. Simon, who showed an appropriate version of the inequality and applied it to prove stabilization in the Allen-Cahn system and in general phase field models, see [20]. Another fifteen year were needed for the appearance of a version of Łojasiewicz inequality which was suitable for an application to Cahn-Hilliard equation, which resulted in another stabilization result, see [19]. Here, we mention only papers which deal with problems related to phase transitions, thus we leave out a huge part devoted to application of Łojasiewicz inequality to evolution problems.

In all the papers mentioned above the nonlinear term was analytic. It turns out that this assumption may be significantly relaxed. The authors of [6] use a version of the inequality for a C^1 functional, which is not analytic and prove stabilization in a non-local phase-field system; other phase field models are studied in [7] and [4]. Interestingly, it is possible to study with the same tools systems with logarithmic singularities, see [1] and the Ginzburg-Landau equations of superconductivity [5].

2 Existence

System (1.1) may be re-written as

$$\begin{aligned} & u_t = D_u \Delta u + f(x,u) - K(\gamma u - b), \quad \text{in } \Omega \times (0,T), \\ & \frac{\partial u}{\partial \mathbf{n}} = 0, \quad \text{on } \partial\Omega \times (0,T), \\ & u(0,x) = u_0(x), \quad x \in \Omega, \end{aligned} \tag{2.1}$$

where the nonlocal operator K is defined as follows. If w given, then $v = -Kw$ is the solution to

$$-D_v \Delta v + av = w, \quad \text{in } \Omega, \tag{2.2}$$

$$\frac{\partial v}{\partial \mathbf{n}} = 0, \quad \text{on } \partial\Omega. \tag{2.3}$$

In this section we shall establish a global in time existence result for smooth initial data. This is achieved in two steps. First, we show local existence. This fact combined with a priori estimates implied by the gradient structure of (2.1) and Lemma 2.1 below yields the global in time existence.

We shall use the language and methods of the semigroup theory. We shall work in the Hilbert space $X_2 = L^2(\Omega)$ as well as in the Banach spaces $X_p = L^p(\Omega)$, $p \in (2,\infty)$. Let us denote by Δ_N the Laplace operator with homogeneous Neumann boundary condition. The operator $D_u(-\Delta_N + 1)$ will play a major role; however for the sake of a compact notation we will denote it by A_p, where p refers to X_p. We remark that the domain of A_p is given by

$$\mathcal{D}(A_p) = \left\{ u \in W^{2,p}(\Omega) : \frac{\partial u}{\partial \mathbf{n}} = 0 \right\}.$$

The operators A_p are sectorial (see [12, Sect. 1.4] and [12, Sect. 1.6]). This fact is particularly easy when $p = 2$, since then A_2 is self-adjoint and positive. Thus, the spaces X_p^α are well-defined as the domains of the operators A_p^α (cf. [12, Chap. 1]). The norm in X_p^α is given by $\|u\|_{\alpha,p} = \|A_p^\alpha u\|_{L^p}$. In the case $\alpha = 1/2$ we denote by $A_p^{1/2}$ the square root of A_p, (cf. [12, Chap. 1]). In fact we can identify $X_p^{1/2}$ with $W^{1,p}(\Omega)$. Indeed, this is particularly easy when

$p=2$, if $u\in\mathcal{D}(D_u(-\Delta_N+1))$; then by the definition of the norm and self-adjointness of Δ_N we have

$$D_u{}^{-1}\|u\|_{1/2,2}^2=\|(-\Delta_N+1)^{1/2}u\|_{L^2}^2=((-\Delta_N+1)u,u)_{L^2}.$$

An integration by parts yields,

$$D_u{}^{-1}\|u\|_{1/2,2}^2=\int_\Omega(|\nabla u|^2+u^2)\,dx=\|u\|_{W^{1,2}}^2. \tag{2.4}$$

As a result we conclude that the norms $\|\cdot\|_{1/2,2}$ and $\|\cdot\|_{W^{1,2}}$ are equivalent, hence we may identify $W^{1,2}(\Omega)$ with $X_2^{1/2}$. For a general result, when $p\in(1,\infty)$, we note that X_p^α is the complex interpolation space $[L^p(\Omega),\mathcal{D}(A_p)]_\alpha$, where $\mathcal{D}(A_p)$ is understood with the graph norm (see [22, Theorem 1.15.3]), moreover by [22, Theorem 4.3.3] we have $[L^p(\Omega),\mathcal{D}(A_p)]_{1/2}=W^{1,p}(\Omega)$.

Now, we introduce a Lyapunov functional coinciding with (1.3) up to a constant, which plays a major role in the study of (1.1). We set

$$E(u)=\int_\Omega\left(\frac{D_u}{2}|\nabla u(x)|^2+F(x,u(x))\right)dx+\frac{1}{2}(\gamma Ku,u)_{L^2}-(Kb,u)_{L^2}+\mathcal{C}, \tag{2.5}$$

where $F(x,u)=F_0-\int_0^u f(x,s)\,ds$, F_0 is a suitable positive number, and $\mathcal{C}$ is chosen to ensure that

$$\frac{1}{2}(\gamma Ku,u)_{L^2}-(Kb,u)_{L^2}+\mathcal{C}\geq 0. \tag{2.6}$$

More precisely we assume that

$$\begin{aligned}
&f(x,u)=-a_3(x)u^3+a_2(x)u^2+a_1(x)u,\\
&a_i,\ i=1,2,3\quad \text{are smooth with all their derivatives bounded in } \Omega,\\
&a_3(x)\geq\delta>0\quad \text{all } x\in\Omega.
\end{aligned} \tag{2.7}$$

Hence we may choose F_0 so that $F(x,u)$ is positive for all $(x,u)\in\Omega\times\mathbb{R}$.

In our example $f(u)=u-u^3$, i.e. $F(u)=\frac{1}{4}(1-u^2)^2$. In view of (2.6) and the definition of F we can choose F_0 large enough so that

$$E(u)\geq\frac{D_u}{2}\|\nabla u\|_{L^2}^2+d\|u\|_{L^4}^4,$$

for some positive constant d.

Moreover, we can check that

$$u_t=-E'(u), \tag{2.8}$$

where E' is the variational derivative of E or more precisely the derivative in the L^2 norm, i.e.

$$E(u+h)-E(u)=(E'(u),h)_{L^2}+o(h),$$

where $h\in W^{1,2}(\Omega)\cap L^4(\Omega)$ and $|o(h)|/\|h\|_{L^2}\to 0$, when $\|h\|_{L^2}\to 0$. Thus,

$$\frac{dE}{dt}(u)=\int_\Omega E'(u)\cdot u_t\,dx=-\|D_u\Delta u+f(\cdot,u)-K(\gamma u-b)\|_{L^2}^2. \tag{2.9}$$

After these preliminary remarks, we state the first existence result. Our goal is to show global existence of smooth solutions, i.e. belonging to $\bigcap_{k=1}^{\infty} \mathcal{D}(A_p^k)$, for some $p > n$, for all $t > 0$. We begin with a local in time result.

Proposition 2.1 *Let us suppose that Ω is an open, bounded subset of $\mathbb{R}^n$ with a smooth boundary, $n \geq 1$ and $p > n$, $N > 0$ are arbitrary. Moreover, $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is smooth and satisfies (2.7) and $F(x, u)$ defined above is positive. We also assume that the initial function u_0 belongs to $\mathcal{D}(A_p^N)$. Then, there exist a positive number T and a unique local in time solution to (2.1), such that*

$$u \in C((0, T]; W^{2N+2,p}(\Omega)) \cap C([0, T]; W^{2N,p}(\Omega)), \qquad u_t \in C((0, T); W^{2N,p}(\Omega)),$$

more precisely, we have that

$$u \in C((0, T]; \mathcal{D}(A_p^{N+1})) \cap C([0, T]; \mathcal{D}(A_p^N)).$$

Proof We will apply the Banach contraction principle. We set $X_T^{2N,p} = C([0, T]; \mathcal{D}(A_p^N))$ with the norm

$$\|u\|_{X_T^{2N,p}} = \sup_{t \in [0,T]} \|A_p^N (u(t))\|_{L^p}.$$

Using the fact that A_p is sectorial we rewrite (2.1) in integral form by means of the variation of constant formula (see [12, Chap. 3])

$$u(t) = e^{-A_p t} u_0 + \int_0^t e^{-A_p (t-s)} (D_u u + f(\cdot, u(s)) - K(\gamma u(s) - b))\, ds. \tag{2.10}$$

If $u \in X_T^{2N,p}$ is given, we denote the right-hand-side of (2.10) by Λu, and we look for a fixed point of Λ.

Let us take a closed ball $B_R \subset X_T^{2N,p}$ centered at $e^{-A_p t} u_0$, i.e. $B_R = \bar{B}(e^{-A_p t} u_0, R)$, $R > 0$. We shall show that for sufficiently small $T > 0$: (a) $\Lambda(B_R) \subset B_R$ and (b) Λ is a strict contraction.

To that purpose we need the following observation, which results from the elliptic regularity theory and the embedding theorems for $p > n$,

$$\|D^j u\|_{L^\infty} \leq C \|A_p^N u\|_{L^p} \tag{2.11}$$

for $j < 2N$.

We first show (a), i.e. that $\Lambda(B_R) \subset B_R$. Suppose that $u \in B_R \subset X_T^{2N,p}$, then $\Delta^N (f(u) + K(\gamma u - b)) \in L^p(\Omega)$. Also using that $\|A_p^\alpha e^{-A_p t}\|_{L^p} \leq C_\alpha t^{-\alpha} e^{-\lambda t}$, where $\lambda > 0$ is the smallest eigenvalue of A_p and $\alpha > 0$ (cf. [12, Theorem 1.4.3]), we deduce

$$\begin{aligned} \|\Lambda u - e^{-A_p t} u_0\|_{X_T^{2N,p}} &= \sup_{t \in [0,T]} \|A_p^N (\Lambda u - e^{-A_p t} u_0)\|_{L^p} \\ &\leq \sup_{t \in [0,T]} \int_0^t C e^{-\lambda (t-s)} \|A_p^N (D_u u + f(\cdot, u(s)) - K(\gamma u(s) - b))\|_{L^p}\, ds. \end{aligned}$$

It is now easy to check that

$$\|A_p^N f(\cdot, u)\|_{L^p} \leq C(N, a_1, a_2, a_3) \|A_p^N u\|_{L^p}^3 \quad \text{and} \quad \|A_p^N K(\gamma u - b)\|_{L^p} \leq C \|A_p^{N-1} u\|_{L^p}.$$

Combining these with $\|u\|_{X_T^{2N,p}} \le \|u - e^{-A_p t} u_0\|_{X_T^{2N,p}} + \|e^{-A_p t} u_0\|_{X_T^{2N,p}}$ we conclude that for u in the ball B_R we have

$$\begin{aligned}\|\Lambda u - e^{-A_p t} u_0\|_{X_T^{2N,p}} \le \sup_{t\in[0,T]} \int_0^t & C e^{-\lambda(t-s)} (C(N, a_1, a_2, a_3)(R + \|u_0\|_{X_T^{2N,p}})^3 \\ & + C(R + \|u_0\|_{X_T^{2N,p}}))\, ds. \end{aligned} \tag{2.12}$$

Thus, we conclude that for sufficiently small T, the operator Λ maps B_R into itself.

(b) After performing similar calculations as those which lead to (2.12) we can see that if $u, v \in B_R$, then

$$\begin{aligned}\|f(\cdot, u) - f(\cdot, v)\|_{L^p} &\le C(a_1, a_2, a_3)(\|u\|^2_{L^\infty} + \|v\|^2_{L^\infty})\|u - v\|_{L^p} \\ &\le C(R^2, \|u_0\|_{X_T^{2N,p}})\|u - v\|_{X_T^{2N,p}},\end{aligned}$$

where we also used that $X_T^{2N,p} \subset L^\infty(Q_T)$. This shows that the mapping

$$X_T^{2N,p} \ni u \mapsto D_u u + f(\cdot, u) - K(\gamma u - b) \in L^p(\Omega)$$

is locally Lipschitz continuous. Hence, by a choice of a sufficiently small $T > 0$ we come to the conclusion that Λ is a strict contraction. This leads to the existence of a unique fixed point. It is now easy to check by using the methods of [12, Sect. 3.2] that the fixed point u not only belongs to $X_T^{2N,p}$, but also to

$$u \in C((0, T]; X_T^{2N+2,p}), \qquad u_t \in C((0, T]; X_T^{2N,p}). \qquad \square$$

In order to prove the global in time existence we need a priori estimates for solutions of (2.10) in $X_T^{2N,p}$ which are independent of time. One estimate is easily available. Indeed, it follows from Proposition 2.1 and (2.9) that $E(u(t)) \le E(u_0)$. By the choice of F_0 this implies that $\|u(t)\|_{L^4} \le C(E(u_0))^{1/4}$ for $t > 0$. Another bound is a version of [2, Lemma 3.3].

Lemma 2.1 *Let $u \in X_T^{2N,p}$, with $p > n$ be the unique solution of (2.1) on $[0, T]$, then for any $p \in [2, \infty)$ we have the bound,*

$$\|u(t)\|_{L^p} \le C(p)(1 + (E(u_0))^{r/4}), \quad \textit{for } t \in [0, \tau],\ \tau = \min\{1, T\}, \tag{2.13}$$

where $r = r(p, n)$ is defined below in (2.17).

Proof We multiply $(1.1)_1$ by $t^\gamma |u|^\alpha u$, where α, γ are to be chosen later, and integrate by parts. By Young inequality we arrive at

$$\begin{aligned}\frac{1}{\alpha+2}\frac{d}{dt}\int_\Omega t^\gamma |u|^{\alpha+2}\, dx &\le \frac{\gamma}{\alpha+2} t^{\gamma-1} \int_\Omega |u|^{\alpha+2}\, dx - t^\gamma \int_\Omega a_3 |u|^{\alpha+4}\, dx \\ &\quad + t^\gamma \int_\Omega (|a_2||u|^{\alpha+3} + |a_1||u|^{\alpha+2})\, dx + t^\gamma \int_\Omega |v||u|^{\alpha+1}\, dx \\ &\le \frac{\gamma}{\alpha+2} t^{\gamma-1} \int_\Omega |u|^{\alpha+2}\, dx - \frac{1}{2} t^\gamma \int_\Omega a_3 |u|^{\alpha+4}\, dx + C(a_1, a_2) \\ &\quad + C_\epsilon t^\gamma \int_\Omega |v|^{(\alpha+4)/3} + \epsilon t^\gamma \int_\Omega |u|^{\alpha+4}\, dx,\end{aligned}$$

where $v = -K(\gamma u - b)$. We note that $t^{\gamma-1}|u|^{2+\alpha} = (t^\gamma |u|^{(2+\alpha)\frac{\gamma}{\gamma-1}})^{\frac{\gamma-1}{\gamma}}$. If we now take $\gamma = \frac{\alpha}{2} + 2$, then we can see that

$$t^{\gamma-1}|u|^{2+\alpha} \le \epsilon t^\gamma |u|^{\alpha+4} + C(\epsilon, \alpha).$$

Thus,

$$\frac{1}{\alpha+2}\frac{d}{dt}\int_\Omega t^\gamma |u|^{\alpha+2}\,dx \le C(\epsilon, \alpha, a_1, a_2)\left(1 + t^\gamma \int_\Omega |v|^{(\alpha+4)/3}\right). \tag{2.14}$$

We now start our iterative process, by setting $q_0 = 4$ and noticing that $u \in L^\infty(0, T; L^{q_0}(\Omega))$. Assuming that $u \in L^\infty(0, T; L^{q_{k-1}}(\Omega))$ we will deduce that $u \in L^\infty(0, T; L^{q_k}(\Omega))$ for properly defined q_k.

If we take into account that v is defined as a solution to (2.2), with data u in $L^{q_{k-1}}$, then by the standard elliptic regularity theory and the embedding $W^{2,q} \subset L^{nq/(n-2q)}$, we conclude that

$$\|v\|_{L^{nq_{k-1}/(n-2q_{k-1})}} \le C\|v\|_{W^{2,q_{k-1}}} \le C\|u\|_{L^{q_{k-1}}}. \tag{2.15}$$

Keeping this in mind we take α in (2.14) satisfying the relation $\frac{\alpha+4}{3} = \frac{nq_{k-1}}{n-2q_{k-1}}$ and we set $q_k := \alpha + 2$. Then we obtain the following recurrent relation

$$q_k = 3\frac{nq_{k-1}}{n - 2q_{k-1}} - 2.$$

We remark that there exists k_0 such that $n \ge 2q_{k_0}$, but $n < 2q_{k_0+1}$. Indeed let us consider three cases, according to the value of the space dimension n. If $n < 8$, then automatically $n < 2q_0 = 8$. If $n = 8$, then we may take for q_1 any positive number, in particular we can require that $2q_1 > n = 8$. Finally, for $n > 9$ and $q_0 \ge 4$ we remark that the sequence $\{q_k\}_{k=0}^\infty$ is strictly increasing as long as $q_{k-1} < n/2$.

Thus, the integration of (2.14) over the interval $[0, t]$, for any $t \le \tau$, and the definition of q_k imply

$$\frac{1}{q_k}\|u(t)\|_{L^{q_k}}^{q_k} \le C(\epsilon, q_k, a_1, a_2)\left(1 + \int_0^t \|v(s)\|_{L^{nq_{k-1}/(n-2q_{k-1})}}^{nq_{k-1}/(n-2q_{k-1})}\,ds\right). \tag{2.16}$$

If we combine (2.15) with (2.16) and the definition of q_k, then we see

$$\|u(t)\|_{L^{q_k}} \le C(\epsilon, q_k, a_1, a_2)\big(1 + \sup_{s\in[0,\tau]} \|u(s)\|_{L^{q_{k-1}}}^{\frac{1}{3}(1+\frac{2}{q_k})}\,ds\big), \quad \text{for } 0 \le t \le \tau.$$

Thus, for $p \in [q_{k-1}, q_k)$, with $k \le k_0$ we iterate this estimate, thus we come to

$$\|u(t)\|_{L^{q_k}} \le C(k, q_k, a_1, a_2)(1 + \max_{t\in[0,\tau]} \|u(t)\|_{L^4}^r) \le C(k, q_k)(1 + E(u_0)^{r/4}),$$

for

$$r = r(p, n) = 3^{-k}\prod_{l=1}^k\left(1 + \frac{2}{q_l}\right), \tag{2.17}$$

where $p \in [q_{k-1}, q_k)$.

If $p > q_{k_0}$, then $\|v\|_{L^\infty} \le \|u\|_{L^{q_{k_0}}}$ and integrating (2.14) over $[0, \tau]$ with $r = 3^{-k_0}\prod_{l=1}^{k_0}(1 + \frac{2}{q_l})$ yields the desired estimate. □

Remark 2.1 Once that we have established (2.13) on $[0,\tau]$ we can extend it to $[0,T]$ for $T\ge 1$. We proceed iteratively on intervals $[k\tau,(k+1)\tau]$. On each of those intervals we may take $u((k+1)\tau)$ in place of u_0 in formula (2.13). Thus Lemma 2.1 implies

$$\|u(t)\|_{L^p}\le C(p)(1+(E(u(k\tau)))^{r/4})\le C(p)(1+(E(u_0))^{r/4})$$

for $t\in[k\tau,(k+1)\tau]$, where k is any positive integer. In the above estimate we also used the fact that E is a Lyapunov functional.

Now, we can iteratively establish the bounds we need.

Lemma 2.2 *Let us fix $p>n$ and a natural number N, $T>0$ and let us suppose that $u\in X_T^{2N,p}$ is a unique solution of* (2.1) *constructed in Proposition* 2.1, *then $\|u(t)\|_{X_T^{2N,p}}\le C(p,n,N,E(u_0),\|u_0\|_{W^{2N,p}})$ for $t\le T$.*

Proof We may apply the operator A_p^η, $\eta\in(\frac12,1)$, to both sides of (2.10) and calculate the L^p norm. This leads us to

$$\|u(t)\|_{W^{1,p}}\le C\|u(t)\|_{\eta,p}\le Ce^{-\lambda t}\|u_0\|_{\eta,p}+\int_0^t C_{\eta,p}\frac{e^{-\lambda(t-s)}}{(t-s)^\eta}\|f(\cdot,u)+K(\gamma u-b)\|_{L^p}\,ds.$$

By the Remark 2.1 above, the nonlinear term is bounded by the data,

$$\|f(\cdot,u)+K(\gamma u-b)\|_{L^p}\le C(1+\|u_0\|_{L^4}^r)\le C(1+E^{r/4}(u_0)),$$

where we have also exploited the choice of F_0 in the definition of E.

Thus, $\sup_{t\in[0,T]}\|u(t)\|_{W^{1,p}}\le C_0(1+E^{r/4}(u_0))$ independently of T. As a result, since $p>n$, we conclude from the embedding theorem that,

$$\sup_{t\in[0,T]}\|u(t)\|_{L^\infty}\le C_1(1+E^{r/4}(u_0))$$

independently of T.

In the next step, we uniformly bound $\|A_pu\|_{L^p}$. Namely, we have

$$\begin{aligned}\|u(t)\|_{W^{2,p}}&\le C\|A_pu(t)\|_{L^p}\\&\le Ce^{-\lambda t}\|u_0\|_{1,p}+\int_0^t C\|A_p^{1/2}e^{-A_p(t-s)}\nabla(f(\cdot,u(s))+K(\gamma u(s)-b)))\|_{L^p}\,ds\\&\le Ce^{-\lambda t}\|u_0\|_{1,p}+\int_0^t C_{1/2,p}\frac{e^{-\lambda(t-s)}}{(t-s)^\eta}\|\nabla(f(\cdot,u(s))+K(\gamma u(s)-b))\|_{L^p}\,ds,\end{aligned}$$

where we have also used the equivalence of the standard norm in $W^{1,p}$ and in $X_p^{1/2}$.

By the previous step the term $\|\nabla f(u)\|_{L^p}$ is bounded in terms of data only and independently of time, because we have such bounds on $\|\nabla u\|_{L^p}$ and $\|u(t)\|_{L^\infty}$. Thus, we get a uniform bound

$$\|u(t)\|_{W^{2,p}}\le C_1(p,E(u_0),\|u_0\|_{W^{2,p}})\quad\text{for } t\in[0,T].$$

This estimate implies by the embedding theorems that $\|\nabla u\|_{L^\infty}$ is uniformly bounded by the data as well.

We shall establish iteratively that

$$\sup_{t\in[0,T]} \|A_p^N u(t)\|_{L^p} \le C(p,n,N,E(u_0),\|u_0\|_{W^{N,p}}), \tag{2.18}$$

which implies the desired bound due to smoothness of u_0.

We have already done it for $N=1$. Let us suppose that this bound holds for a number $l\ge 1$, we will show it for $l+1$. For this purpose we apply the operator $A_p^{l+1/2}$ to both sides of (2.10), thus

$$\begin{aligned}\|A_p^{l+1/2}u(t)\|_{L^p} &\le Ce^{-\lambda t}\|u_0\|_{l+1/2,p}\\ &\quad+\int_0^t \|A_p^{1/2}e^{-A_p(t-s)}A_p^l(f(\cdot,u(s))+K(\gamma u(s)-b))\|_{L^p}\,ds.\end{aligned}$$

In order to proceed we make the observation that if $p>n$ and $a\in C^\infty(\bar\Omega)$, then $D^M(au^3)\in L^p$ and

$$\|D^M(au^3)\|_{L^p}\le C(a,M)\|u\|^3_{W^{M,p}}. \tag{2.19}$$

To this end we notice that $D^M(au^3)$ is a sum composed of the terms $D^iaD^juD^kuD^mu$, where $j+k+m\le M$. Due to boundedness of D^ia, it is sufficient to show that each of these products belongs to L^p. By Hölder inequality we have the following bound

$$\begin{aligned}&\int_\Omega |D^juD^kuD^mu|^p\,dx\\ &\quad\le\left(\int_\Omega |D^ju|^{p\alpha_j}\,dx\right)^{1/\alpha_j}\left(\int_\Omega |D^ku|^{p\alpha_k}\,dx\right)^{1/\alpha_k}\left(\int_\Omega |D^mu|^{p\alpha_m}\,dx\right)^{1/\alpha_m},\end{aligned}$$

where $\frac{1}{\alpha_j}+\frac{1}{\alpha_k}+\frac{1}{\alpha_m}=1$. The exponents $p\alpha_j$, $p\alpha_k$, $p\alpha_m$ must be no greater than the exponents arising from the Sobolev embeddings. We note that $D^ru\in W^{M-r,2}$ (recall $p\ge 2$), thus $D^ru\in L^{p_{M,r}}$, where $p_{M,r}=\frac{pn}{n-p(M-r)}$. It sufficient to check that $\frac{1}{p_{N,j}}+\frac{1}{p_{N,k}}+\frac{1}{p_{N,l}}\le\frac{1}{p}$. A direct calculation shows that this is the case. Moreover, combining the inequalities above for all j,k,m such that $j+k+m\le M$ we conclude that (2.19) holds for any M.

Having (2.19) at our disposal we conclude that

$$\|A_p^{l+1/2}u(t)\|_{L^p}\le Ce^{-\lambda t}\|u_0\|_{l+1/2,p}+\int_0^t C_{2,p}e^{-\lambda(t-s)}C_l(p,E(u_0),\|u_0\|_{W^{2l,p}}). \tag{2.20}$$

In the next step we apply A_p^{l+1} to both sides of (2.10); proceeding as above we arrive at the estimate

$$\begin{aligned}\|A_p^{l+1}u(t)\|_{L^p}&\le Ce^{-\lambda t}\|u_0\|_{l+1,p}\\ &\quad+\int_0^t \|A_p^{1/2}e^{-A_p(t-s)}A_p^{l+1/2}(f(\cdot,u(s))+K(\gamma u(s)-b))\|_{L^p}\,ds.\end{aligned}$$

At this point we recall that the norms $\|\cdot\|_{1/2,p}$ and $\|\cdot\|_{W^{1,p}}$ are equivalent. An application of (2.19) and (2.20) yields the desired result (2.18).

The bound (2.18) and the method used above imply that the following estimate hold

$$\sup_{t\in[0,T]} \|A_p^{N+1/2}u(t)\|_{L^p} \le \sup_{t\in[0,T]} C\|\nabla A_p^N u(t)\|_{L^p} \le C\left(p, n, N+\frac{1}{2}, E(u_0), \|u_0\|_{W^{N,p}}\right). \tag{2.21}$$

□

This fact yields a global in time solution.

Theorem 2.1 *Let us suppose that Ω and f are as in Proposition 2.1. In addition we assume that F is positive and $u_0 \in \bigcap_{k=1}^{\infty} \mathcal{D}(A_p^k)$, where $p > n$ is arbitrary. Then, the solution constructed in Proposition 2.1 is global in time. Moreover, $\|u(t)\|_{W^{2N,p}} \le M_N$, for any $N > 1$, for all $t > 0$, where M_N is independent of time.*

Proof By the preceding lemmas we deduce that $\|u\|_{X_T^{2N,p}}$ stays bounded independently of T,

$$\sup_{t\in(0,T)} \|u(t)\|_{W^{2N,p}} \le C(p, n, N, E(u_0), \|u_0\|_{W^{2N,p}}).$$

This fact, (2.21) and the method used to establish [12, Theorem 3.3.4] imply that the limit

$$\lim_{t\to T^-} u(t)$$

exists in $W^{2N,p}(\Omega)$. Thus, we may extend the solution to a maximal interval of existence $[0, T_\infty)$. But the above bound which is uniform in time implies that $T_\infty = \infty$. □

3 Asymptotic Behavior

In order to establish the existence of the ω-limit set, we first show the precompactness of the orbit. First we notice that the ω-limit set may only consist of steady states, because of the fact that (2.1) is a gradient system, see (2.9).

Proposition 3.1 *Let us suppose that the assumptions of Theorem 2.1 hold. Then, for any natural number $N \ge 1$, the set $\omega(u_0)$ is compact in $H^{2N}(\Omega)$ and connected; moreover it only consists of the stationary points of (2.1) and E is constant on $\omega(u_0)$.*

Proof We have already shown that the set $\{u(t) : t \in [0,\infty)\}$ is bounded in $W^{2N+2,p}(\Omega)$. The existence of a compact in $H^{2N}(\Omega)$ connected omega-limit set follows from the fact that $p \ge 2$. Connectedness of $\omega(u_0)$ follows immediately from the definition of this set, see [12, Theorem 4.3.3]. Since (2.1) is a gradient system, and E is its Lyapunov function, this implies that only stationary points may belong to $\omega(u_0)$. We can also infer from the fact that E decreases along the trajectories that E must be constant on $\omega(u_0)$. □

We are now ready to state the main result of this paper.

Theorem 3.1 *Let us suppose that Ω is a bounded region of $\mathbb{R}^n$ with smooth boundary, and that f satisfies the assumption of Proposition 2.1. We also assume that the initial datum u_0 of (2.1) belongs to $\bigcap_{k=1}^{\infty} \mathcal{D}(A_p^k)$ (hence it is smooth). Then, the unique solution to (2.1) converges to a stationary state in $H^N(\Omega)$, for all $N \in \mathbb{N}$, $N \ge 1$, as time goes to infinity.*

Our method of proof is based on the Łojasiewicz inequality and on the results presented by Chill [3]; more specifically we will use [3, Theorem 2]. First, we recall the setting used in [3]. Namely, we suppose that V and H are two Hilbert spaces, such that V is continuously and densely embedded into H. We denote by H^* the dual space of H. Let the functional $E : V \to \mathbb{R}$ be twice continuously differentiable, $E \in C^2(V)$ and denote by L the second derivative E''. Further, let $\varphi \in V$ be a critical point of E, so that $E'(\varphi) = 0$. If $P : H \to H$ is the orthogonal projection onto $\ker L(\varphi)$, then one can define the critical manifold S,

$$S = \{u \in V : (I - P)E'(u) = 0\}.$$

We recall.

Lemma 3.1 [3, Lemma 1] *We assume that $E \in C^2(V)$, $\varphi \in V$ is a critical point of E and $E''(\varphi) = L(\varphi)$ is a Fredholm operator, i.e. the kernel and the orthogonal complement of the image of E'' are finite dimensional spaces. Then, the set S is, locally near φ, a differentiable manifold such that*

$$\dim S = \dim \ker L(\varphi).$$

If $E \in C^k(V)$, $k \geq 2$, then S is a C^{k-1}-manifold. If E is analytic, then S is analytic.

The main result of [3] is the following.

Proposition 3.2 [3, Theorem 2] *Let us suppose that the assumptions of the above Lemma hold. In addition we assume that $E|_S$ satisfies the Łojasiewicz inequality near φ, namely that there exist a neighborhood $U \subset V$ of φ and a constant $\theta \in (0, \frac{1}{2}]$, such that*

$$|E(u) - E(\varphi)|^{1-\theta} \leq C\|E'(u)\|_{V^*}, \quad \textit{for every } u \in U \cap S.$$

Then E itself satisfies the Łojasiewicz inequality in an open set W of V containing φ, with the same Łojasiewicz exponent θ.

We will make a suitable choice of H and V and prove that the hypotheses of this proposition are satisfied. This will be done in the course of the proof of our main result.

Proof of Theorem 3.1 Let us suppose that N is arbitrary but larger than $n/2$, so that $H^N(\Omega)$ is continuously embedded into $C(\bar{\Omega})$. We noticed in Proposition 3.1 that $\omega(u_0)$ only consists of stationary points of (2.1) and that there exists a constant e such that

$$E(u) = e, \quad \text{for all } u \in \omega(u_0). \tag{3.1}$$

We have to check that the assumptions of Proposition 3.2 are satisfied. We choose $H = L^2(\Omega)$ and $V = H^N(\Omega)$. Since $L(\varphi)$ corresponds to the linearization of (2.1), it is a Fredholm operator. Indeed, $L(\varphi)$ is a sum of the Laplace operator on domain $\mathcal{D}(A_2)$, which makes it a self-adjoint operator, and a bounded linear self-adjoint operator on $L^2(\Omega)$. It follows that L is self-adjoint too, hence its kernel and co-kernel coincide. Moreover, $L(\varphi)$ is a strongly elliptic operator and the boundary of Ω is smooth, so its kernel is finite dimensional, because in such a case all the eigenspaces are finite dimensional. The analyticity of the functional E, defined by (2.5), follows from the fact that f is a polynomial in u. Hence, the critical manifold S is analytic due to Lemma 3.1. As a result Łojasiewicz inequality holds for E restricted to S (see [13, 15, Sect. IV.9]) and due to Proposition 3.2 it is true also in V, i.e. if φ is a critical point of E then there exist $\beta > 0$ and $\theta \in (0, 1/2)$ such that

$$|E(u) - E(\varphi)|^{1-\theta} \le C\|E'(u)\|_{(H^N)^*}, \quad \text{for } \|u - \varphi\|_{H^N} \le \beta.$$

However, since (1.1) is a gradient flow in the L^2 norm we have to obtain an upper bound on $\|E'(u)\|_{(H^N)^*}$ in terms of $\|E'(u)\|_{L^2}$. Indeed, by the definition of the norm in the adjoint space, we have

$$\begin{aligned}\|E'(u)\|_{(H^N)^*} &= \sup_{\varphi\in H^N} \frac{\langle E'(u), \varphi\rangle}{\|\varphi\|_{H^N}} \\ &= \sup_{\varphi\in H^N} \frac{1}{\|\varphi\|_{H^N}}\left(\int_\Omega (\nabla u \nabla\varphi - f(x,u)\varphi)\,dx + \gamma(Ku,\varphi)_{L^2} - (Kb,\varphi)_{L^2}\right).\end{aligned}$$

Now, integration by part and the Cauchy inequality yield,

$$\|E'(u)\|_{(H^N)^*} \le \sup_{\varphi\in H^N} \frac{1}{\|\varphi\|_{H^N}}(\|-\Delta u - f(x,u) + \gamma Ku - Kb\|_{L^2}\|\varphi\|_{L^2}).$$

Finally, we obtain,

$$|E(u) - E(\varphi)|^{1-\theta} \le C_0\|E'(u)\|_{L^2}, \quad \text{for } \|u - \varphi\|_{H^N} \le \beta, \tag{3.2}$$

where e is defined by (3.1).

In view of the compactness of $\omega(u_0)$ in H^N there exists $\mathcal{U}$ a neighborhood of $\omega(u_0)$, composed of a finite number of balls B_j, $j = 1, \dots, N_{\omega(u_0)}$. In each of the balls B_j inequality (3.2) holds with an exponent θ_j and a constant C_j. We take a common exponent $\bar\theta = \min\{\theta_j : j = 1, \dots, N_{\omega(u_0)}\}$ and a common constant $C = \max\{C_j : j = 1, \dots, N_{\omega(u_0)}\}$ so that we have

$$|E(u) - e|^{1-\bar\theta} \le \bar C\|E'(u)\|_{L^2}, \quad \text{for } u \in \mathcal{U}. \tag{3.3}$$

Moreover, since the distance from $u(t)$ to the ω-limit set converges to zero (see [11, Sect. 3.1], [12, Theorem 4.3.3]), we deduce that there exists a positive constant T such that for all $t > T$, $u(t) \in \mathcal{U}$. Hence, by (2.8)

$$-\frac{d}{dt}|E(u) - e|^{\bar\theta} = -\bar\theta|E(u) - e|^{\bar\theta-1}\langle E'(u), u_t\rangle = \bar\theta|E(u) - e|^{\bar\theta-1}\|E'(u)\|_{L^2}\|u_t\|_{L^2}.$$

Now, the application of (3.3) yields the integrability of $\|u_t\|_{L^2}$,

$$-\frac{d}{dt}|E(u) - e|^{\bar\theta} \ge \frac{\bar\theta}{\bar C}\|u_t\|_{L^2}.$$

Hence, $u(t)$ satisfies the Cauchy condition, i.e. for any $\epsilon > 0$, there is $t_\epsilon > 0$ so that for all $t_1 > t_2 > t_\epsilon$ we have

$$\|u(t_1) - u(t_2)\|_{L^2} \le \int_{t_2}^{t_1} \|u_t(s)\|_{L^2} \le \frac{\bar C}{\bar\theta}(|E(u(t_2)) - e|^{\bar\theta} - |E(u(t_1)) - e|^{\bar\theta}) < \epsilon. \tag{3.4}$$

In the first inequality above we used the formula $u(t_1) - u(t_2) = \int_{t_2}^{t_1} u_t(s)\,ds$ and the triangle inequality.

Since $E(u(t))$ is bounded below and decreases along the orbit, it follows that $E(u(t))$ converges to e as t goes to infinity. Therefore, the right-hand-side of (3.4) can be made arbitrarily small by taking t_1 and t_2 large enough. Hence, $u(t)$ is a Cauchy sequence, thus it converges in L^2 to a stationary solution as $t \to \infty$ and the convergence takes place in the H^N-topology, with $N > n/2$ arbitrary. □

Remark 3.1 It is known that if the solution converges to an isolated stationary point, then the rate of convergence is exponential, see e.g. [3].

The result proven in Theorem 3.1 also extends to $f(x,u)$ being a polynomial in u of odd degree with smooth coefficients, such that the coefficient of highest degree monomial is strictly negative, namely

$$f(x,u) = \sum_{l=1}^{2q-1} a_l(x) u^l,$$

where $a_l \in C^\infty(\bar{\Omega})$, and $a_{2q-1} \leq -\delta < 0$. This is indeed so, since the proof of the key estimate in Lemma 2.1 extends to this case (cf. [2, Lemma 3.3]). The details are left to the reader.

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